Parametrized Surface

A parametrized surface in $\mathbb{R}^3$ can be described locally by the vector-valued C^1 function

$$X(u,v) = [\alpha(u,v), \beta(u,v), \gamma(u,v)].$$

A possible definition of the normal N at point $[x_0, y_0, z_0]$ corresponding to parameter values (u_0, v_0) is

$$N = N(u_0,v_0) = [\frac{\partial(\beta,\gamma)}{\partial(u,v)}, -\frac{\partial(\alpha,\gamma)}{\partial(u,v)}, \frac{\partial(\alpha,\beta)}{\partial(u,v)}]\Big|_{(u_0,v_0)}.$$

Let's verify that this agrees with another of our definitions, by computing and using the implicit function theorem. Suppose

$$\frac{\partial(\alpha,\beta)}{\partial(u,v)} \neq 0, \text{at } u_0, v_0.$$

Consider the equations

$$F(x,y,u,v) = x - \alpha(u,v) = 0,\ G(x,y,u.v) = y - \beta(u,v) = 0.$$

By the preceding line we can solve for u, v near (x_0, y_0, u_0, v_0) as C^1 functions $u = u(x,y), v = v(x,y)$. Using the chain rule we find

$$u_x = \frac{\beta_v}{\frac{\partial(\alpha,\beta)}{\partial(u,v)}},\ v_x = -\frac{\beta_u}{\frac{\partial(\alpha,\beta)}{\partial(u,v)}}.$$

Also using the implicit function theorem, we see that locally the points on the parametrized surface can be written as the graph of a C^1 function

$$z = f(x,y),\ \text{where } f(x,y) = \gamma(u(x,y), v(x,y)).$$

In this form a normal to the surface can be written as

$$n = [-f_x, -f_y, 1].$$

Let's see if this is a non-zero multiple of N. To see this we use the chain rule again to first of all compute

$$f_x = \gamma_u u_x + \gamma_v v_x,$$

and then use previous formulas for u_x and v_x to get

$$-f_x = \frac{\frac{\partial(\beta,\gamma)}{\partial(u,v)}}{\frac{\partial(\alpha,\beta)}{\partial(u,v)}},\ -f_y = -\frac{\frac{\partial(\alpha,\gamma)}{\partial(u,v)}}{\frac{\partial(\alpha,\beta)}{\partial(u,v)}}.$$

You have to *very* careful to get the signs right. If you subsitute this into the formula for n and clear denominators, you get the formula for N. **Always** use the chain rule.